

# 7. TRIGONOMETRIC FUNCTIONS

## §7.1. Elementary Trigonometry

In this section we revise the elementary trigonometry that you learnt in High School. It concerns angles and triangles, especially right-angled ones.

Angles are traditionally measured in degrees, where 360 degrees constitute a complete revolution. This is a legacy from the Babylonians who used a number system based on base 60. The other



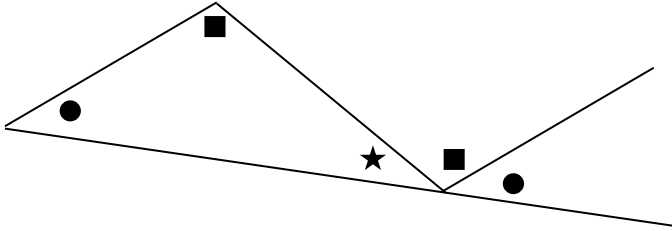
measure that comes from the Babylonians is time where there are 60 minutes in an hour and 60 seconds in a minute.

A **straight angle** is the angle at any point on a straight line. It's half a complete revolution and so is 180 degrees.

A perpendicular angle, or a **right angle**, has 90 degrees. We write this as  $90^\circ$ . A right-angled triangle is one where one angle is a right angle. The other two angles add up to  $90^\circ$  and are called **complementary angles**. The following proof is what one could call “a proof without words”.

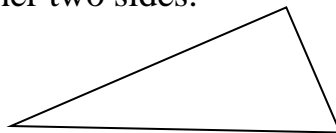
**Theorem 1:** The angles of a triangle total  $180^\circ$ .

**Proof:**



Well, perhaps a few words are called for. The line sticking up at the right is supposed to be parallel to one of the sides of the triangle. We use a couple of theorems from Euclidean Geometry. When a transversal cuts two parallel lines, corresponding angles, marked ●, are equal and alternate angles, marked ■, are equal. 🙌😊

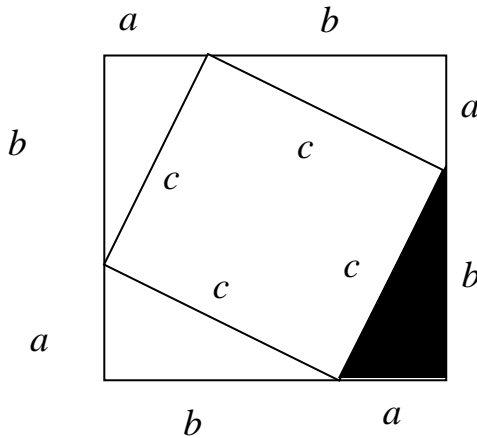
The longest side of a right-angled triangle, the side opposite the right angle, is called the **hypotenuse**. The fundamental fact about right-angled triangles is the theorem that is attributed to Pythagoras: The square of the length of the hypotenuse is the sum of the squares on the lengths of the other two sides.



hypotenuse

**Theorem 2 (Pythagoras):** If the lengths of the sides of a right-angled triangle are  $a$ ,  $b$ ,  $c$  and  $c$  is the length of the hypotenuse then  $a^2 + b^2 = c^2$ .

**Proof:**

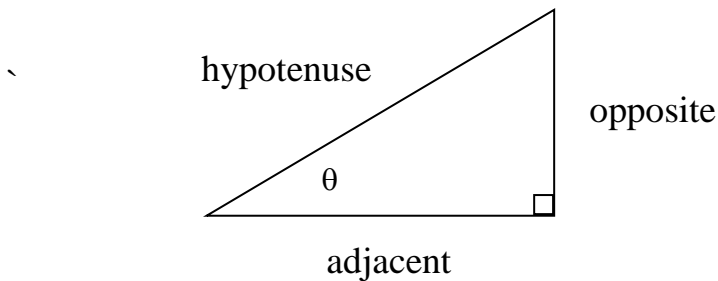


The areas of the larger square is:

$$(a + b)^2 = c^2 + 4 \cdot \frac{1}{2} \cdot ab = c^2 + 2ab.$$

Hence  $a^2 + b^2 = c^2$ . 🙌😊

Suppose we have a right-angled triangle where the angles, in degrees are  $90$ ,  $\theta$  and  $90 - \theta$ .



If we focus on the angle  $\theta$  then the **opposite side** is, as the name suggests, the side opposite that angle. The hypotenuse is the side opposite the right-angle and the remaining side is called the **adjacent side**.

The sides of the triangle will change as the triangle gets bigger or smaller, with the same angle  $\theta$ . But, since corresponding sides of similar triangles are proportional the ratio of corresponding sides will depend only on the angle.

We define the **sine** of  $\theta$  to be  $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ ,

the **cosine** of  $\theta$  is defined as  $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$  and

the **tangent** of  $\theta$  is defined as  $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opposite}}{\text{adjacent}}$ .

A consequence of Pythagoras' Theorem is that:

$$\sin^2\theta + \cos^2\theta = 1.$$

[Here we write  $\sin^2\theta$  when we mean  $(\sin \theta)^2$ . Similar comments apply to  $\cos^2\theta$  and  $\tan^2\theta$ .]

The three other ratios are the reciprocals of the above.

The **secant** of  $\theta$  is  $\sec \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ ,

the **cosecant** of  $\theta$  is  $\text{cosec } \theta = \frac{\text{hypotenuse}}{\text{opposite}}$  and

the **cotangent** of  $\theta$  is  $\cot \theta = \frac{\text{adjacent}}{\text{opposite}}$ .

Rather than burden our brains with these definitions we remember that:

$$\sec \theta = \frac{1}{\cos \theta}, \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad \text{and} \quad \cot \theta = \frac{1}{\tan \theta}.$$

Simple consequences of these definitions are the trigonometric identities:

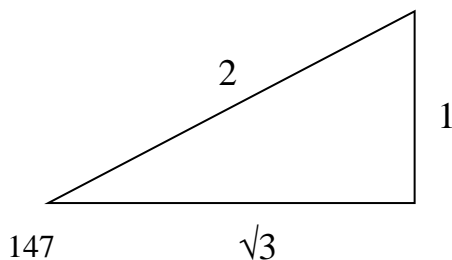
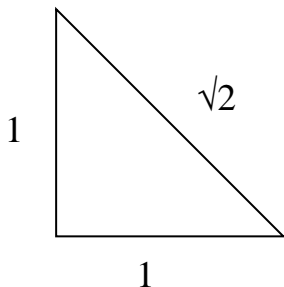
$$\sec^2 \theta = 1 + \tan^2 \theta \quad \text{and} \quad \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta.$$

The first of these is important enough to be memorised but the second is not. It can always be worked out when needed.

A general principle with an area of mathematics, like trigonometry, with many formulae is to remember only a key few and to be able to work out any others that might be needed as the occasion arises.

## §7.2. Special Angles

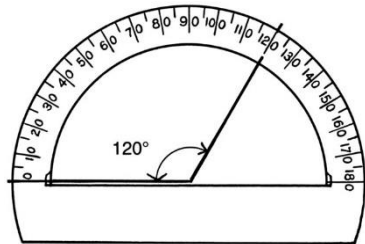
There are two right-angled triangles that are rather special: the 45 degree triangle and the 60-30 triangle. Indeed these are the two standard shapes for set squares.



The dimensions can be obtained from the fact that a 45 degree right-angled triangle is isosceles, a 60-30 triangle is half an equilateral triangle and, of course, Pythagoras' Theorem. Although a right-angled triangle with two right angles and one zero angle is not a proper triangle we can infer the trigonometric functions for  $0^\circ$  and  $90^\circ$ . These are included in the following table.

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
<b>sin</b>	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
<b>cos</b>	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
<b>tan</b>	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\infty$

The last value is a bit of a cheat because there is no such number as infinity. It reflects the fact that if  $\theta$  is very close to  $90^\circ$  the adjacent side is very short and so  $\tan \theta$  is very large. In fact we can make  $\tan \theta$  as large as we like by making  $\theta$  sufficiently close to  $90^\circ$ . Usually we capture this property by saying that as  $\theta$  approaches  $90^\circ$ ,  $\tan \theta$  approaches infinity.



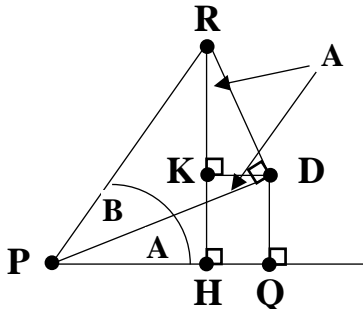
### §7.3. Sum And Difference of Angles

**Theorem 3:**  $\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$

$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$

$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

**Proof:** We'll prove this only in the case where the angles  $A$ ,  $B$  are acute, that is, less than  $90^\circ$ . In the following diagram we have four right-angled triangles. In some of them there is an angle  $A$ , in one of them there's an angle  $B$  and in another there's an angle  $A + B$ .



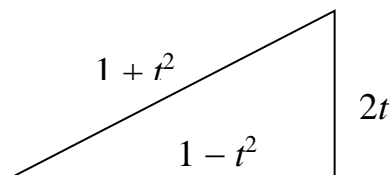
$$\begin{aligned} \sin(A + B) &= \frac{RH}{RP} = \frac{KH + RK}{RP} \\ &= \frac{KH}{RP} + \frac{RK}{RP} \\ &= \frac{KH}{PD} \cdot \frac{PD}{RP} + \frac{RK}{RD} \cdot \frac{RD}{RP} \\ &= \frac{DQ}{PD} \cdot \frac{PD}{RP} + \frac{RK}{RD} \cdot \frac{RD}{RP} \quad \text{since } KH = DQ \\ &= \sin(A) \cos(B) + \cos(A) \sin(B). \end{aligned}$$

$$\begin{aligned}
\cos(A + B) &= \frac{PH}{RP} = \frac{PQ - HQ}{RP} \\
&= \frac{PQ}{RP} - \frac{HQ}{RP} \\
&= \frac{PQ}{PD} \cdot \frac{PD}{RP} - \frac{HQ}{RD} \cdot \frac{RD}{RP} \\
&= \frac{PQ}{PD} \cdot \frac{PD}{RP} - \frac{KD}{RD} \cdot \frac{RD}{RP} \\
&= \cos(A) \cos(B) - \sin(A) \sin(B).
\end{aligned}$$

$$\begin{aligned}
\tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} \\
&= \frac{\sin(A) \cos(B) + \cos(A) \sin(B)}{\cos(A) \cos(B) - \sin(A) \sin(B)} \\
&= \frac{\tan(A) + \tan(B)}{1 - \tan(A) \tan(B)} \quad \text{after dividing top and} \\
&\quad \text{bottom by } \cos(A) \cos(B). \quad \text{👉😊}
\end{aligned}$$

**Corollary:**  $\sin(2A) = 2\sin(A) \cos(A)$ ,  
 $\cos(2A) = \cos^2(A) - \sin^2(A)$  and  
 $\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$ .

The above identities are usually called the **Double Angle Formulae**. The following are the **Half Angle Formulae** where  $\sin(A)$ ,  $\cos(A)$  and  $\tan(A)$  are all expressed in terms of  $t = \tan(A/2)$ .



**Theorem 4:** If  $t = \tan(A/2)$  then

$$\sin(A) = \frac{2t}{1+t^2}, \quad \cos(A) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \tan(A) = \frac{2t}{1-t^2}.$$

**Proof:** The third of these is simply the Double Angle Formula, using  $A/2$  in place of  $A$ . By drawing a suitable right-angled triangle, with sides  $2t$  and  $1-t^2$  we can deduce that the hypotenuse is  $1+t^2$  and hence read off  $\sin(A)$  and  $\cos(A)$ . 🙌😊

**Theorem 5:**  $\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$ ,  
 $\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$

and

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)}.$$

**Proof:** This is proved similarly using the diagram from Theorem 3.

Let  $C = A + B$ . Then by similar methods to those used in the proof of Theorem 2 we get:

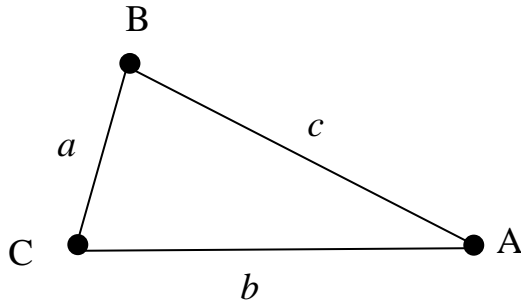
$$\sin(C - B) = \sin(C) \cos(B) - \cos(C) \sin(B)$$

which is what we have to prove but with different notation. The proofs of the other identities are left as exercises. 🙌😊

## §7.4. Triangle Identities

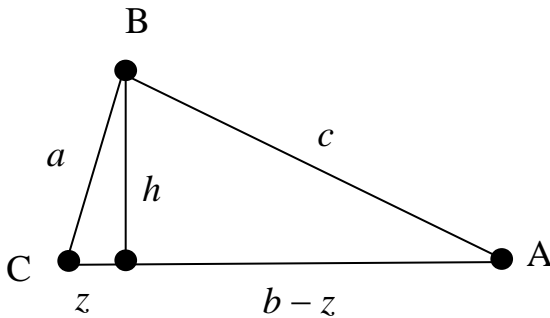
We now move away from right-angled triangles and consider general triangles. We'll label the three vertices  $A, B, C$  and also use the symbols  $A, B, C$  for the values

of the corresponding angles. The sides opposite these sides will be labelled  $a, b, c$  respectively.



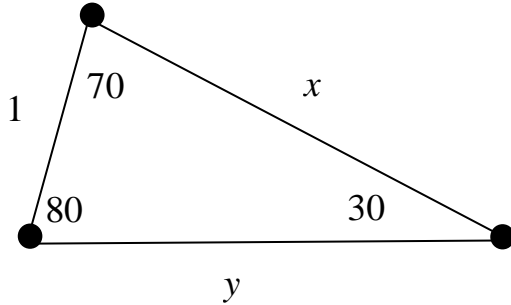
**Theorem 6 (Sine Rule):**  $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ .

**Proof:** Drop a perpendicular from B to the side AC.



Then  $h = a \cdot \sin C = c \cdot \sin A$ . By drawing perpendiculars to the other two sides the result follows. 🙌😊

**Example 1:** Find the remaining three sides of the following triangle:



**Solution:**  $\frac{\sin 80}{x} = \frac{\sin 30}{y} = \frac{1/2}{10} = \frac{1}{20}$ .

Hence  $x = 20 \sin 80 = 20 \times 0.9848 = 19.6960$  and  $\frac{y}{\sin 70} = 20$  so  $y = 20 \sin 70 = 20 \times 0.9397 = 18.7940$ .

**Theorem 7 (COSINE RULE):**

$$c^2 = a^2 + b^2 - 2ab.\cos C.$$

**Proof:**  $z = a.\cos C$  and  $h = a.\sin C$ .

Hence, by Pythagoras in the right-angled triangle,

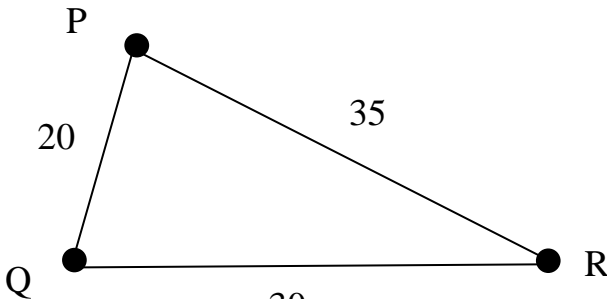
$$\begin{aligned} c^2 &= (b - a.\cos C)^2 + a^2.\sin^2 C \\ &= b^2 + a^2.\cos^2 C - 2ab.\cos C + a^2.\sin^2 C \\ &= a^2 + b^2 - 2ab.\cos C. \end{aligned}$$

This can be used to find the angles of a triangle if we are just given the sides:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

It's probably best to remember the Cosine Rule as “the square on one side is the sum of the squares on the other two sides minus twice the product of the other two sides and the cosine of the included angle”. This way, if the notation varies you can easily adapt the rule.

**Example 2:** Find the three angles of the following triangle:



**Solution:**  $\cos P = \frac{20^2 + 35^2 - 30^2}{2 \cdot 20 \cdot 35} = \frac{725}{1400} = 0.5179.$

Hence  $P = 58.8^\circ.$

$$\cos Q = \frac{20^2 + 30^2 - 35^2}{2 \cdot 20 \cdot 30} = \frac{75}{1200} = 0.0625.$$

So  $Q = 86.4^\circ.$

Hence  $R = 180^\circ - 58.8^\circ - 86.4^\circ = 34.8^\circ.$

(Of course we could have used the cosine rule one more time to find R and use this as a check.) If you don't know how to get from  $\cos P = 0.5179$  to  $P = 58.8^\circ$  using your calculator look ahead to §7.7.

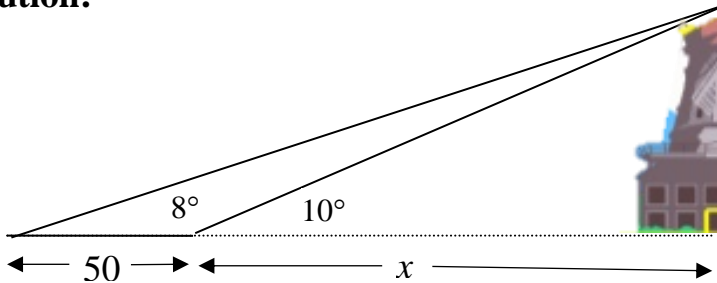
These formulae are useful, in many cases, for finding the remaining sides and angles given three pieces of information. This can be summarised in the following table.

KNOWN	RULE
3 sides	Cosine Rule
2 sides and included angle	Cosine Rule
2 sides and a non-included angle	Not unique
1 side and 2 angles (really 3)	Sine Rule
3 angles	Need a side

**Example 3:** Suppose you want to determine the height of a windmill on a small island in a river. You stand on the river bank and determine that the angle of elevation is  $10^\circ$ . If you could measure your distance from the base of the windmill you could use simple trigonometry to calculate the height of the windmill. But with the river in between this is not feasible. Suppose you walk back 50 metres further away from the windmill, and find that the angle of elevation is  $8^\circ$ . How high is the windmill?



**Solution:**



Suppose the height of the windmill is  $h$  and the distance from the first observation point to the base of the windmill is  $x$ .

$$\text{Hence } \tan 8 = \frac{h}{x + 50} \text{ and } \tan 10 = \frac{h}{x}.$$

$$\text{So } x + 50 = \frac{h}{\tan 8} \text{ and } x = \frac{h}{\tan 10}.$$

$$\text{and } \frac{h}{\tan 10} + 50 = \frac{h}{\tan 8}.$$

$$\text{Therefore } h \left( \frac{1}{\tan 8} - \frac{1}{\tan 10} \right) = 50.$$

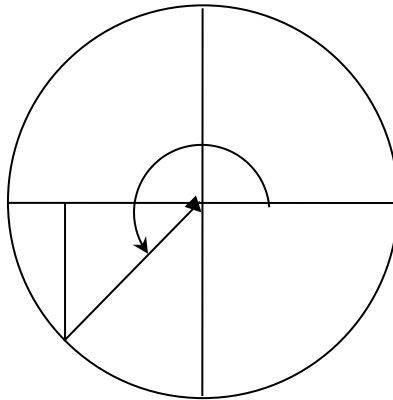
$$\text{Hence } 1.44h = 50 \text{ and so } h = \frac{50}{1.44} = 34.6.$$

Hence the windmill is 34.6 metres high.

## §7.5. Rotations in the $x$ - $y$ Plane

So far angles have been restricted to the range  $0^\circ$  to  $90^\circ$ . We now extend the definition of  $\sin\theta$ , etc for all  $\theta$ .

We do this by considering the unit circle (that is, a circle of radius 1) and rotations in the  $x$ - $y$  plane.



We begin with the positive  $x$ -axis and measure rotations as positive in the anti-clockwise direction and negative in the clockwise direction. The rotation indicated above is positive and its magnitude is about  $225^\circ$ . If this rotation were to be carried out clockwise it would have been through an angle of  $135^\circ$ . If this clockwise rotation had been carried out twice it would have been through an angle of  $270^\circ$ .

If the point  $(1, 0)$ , on the positive half of the  $x$ -axis, is rotated through an angle  $\theta$ , the resulting point is defined to be  $(\cos \theta, \sin \theta)$ . This now extends the definition of  $\sin \theta$  and  $\cos \theta$  to all  $\theta$ . For example, as the above diagram shows, if the point  $(1, 0)$  is rotated through  $225^\circ$  the resulting point is  $(-1/\sqrt{2}, 1/\sqrt{2})$ . (That is because the right-angled triangle shown has two equal sides and its

hypotenuse is 1. Hence the length of each side is  $1/\sqrt{2}$ . It follows that  $\cos 225^\circ = -1/\sqrt{2}$  and  $\sin 225^\circ = 1/\sqrt{2}$ .

If we rotated  $(1, 0)$  through  $585^\circ$ , that is  $225^\circ$  plus  $360^\circ$ , the resulting point would still be  $(-1/\sqrt{2}, -1/\sqrt{2})$ . An extra  $360^\circ$  in either direction does not make any difference to the position of the rotated point and therefore makes no difference to the values of  $\sin \theta$  and  $\cos \theta$ .

**Theorem 8:** For all angles,  $\theta$ , measured in degrees:

$$\sin(\theta \pm 360) = \sin \theta$$


$$\cos(\theta \pm 360) = \cos \theta,$$

$$\sin(\theta + 180) = -\sin \theta$$

$$\cos(\theta + 180) = -\cos \theta,$$

$$\sin(\theta + 90) = \cos \theta$$

$$\cos(\theta + 90) = -\sin \theta.$$

**Proof:** These can easily be proved by considering the  $x$ - $y$  plane. 

The absolute values of  $\sin \theta$  and  $\cos \theta$  lie between 0 and 1 but in different quadrants they have different signs. By considering the  $x$ - $y$  plane we can see that these signs are as follows:

$\sin \theta > 0$	$\sin \theta > 0$
$\cos \theta < 0$	$\cos \theta > 0$
$\sin \theta < 0$	$\sin \theta < 0$
$\cos \theta < 0$	$\cos \theta > 0$

We defined  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  and so  $\tan (\theta + 180) = -\tan \theta$  etc. The value of  $\tan \theta$  is positive in the 1<sup>st</sup> and 3<sup>rd</sup> quadrants and negative in the others. A useful diagram, to help you remember in which quadrants  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$  are positive is the following.

S	A
T	C

Here A stands for ALL, S stands for SIN, T stands for TAN and C stands for COS. Students are often taught a mnemonic “All Stations To ...” where C is represented by some local railway station that starts with “C”. As I learnt this at Canterbury Boys High School, naturally our teachers taught us to remember “All Stations To Canterbury.”

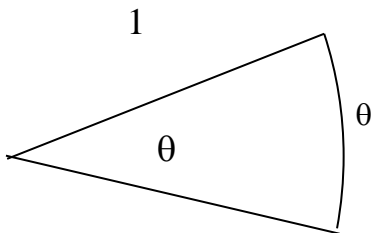
Since most of our proofs of trigonometric identities relied on diagrams where angles were in the range 0 to 90 we should provide new proofs to guarantee that they work for all angles. We will not do that because we are not yet at the final stage of our development of trigonometry.

## §7.6. Radian Measure

Degrees are a very suitable unit of measure for angles. Other systems have been proposed such as the one that assigns 100 ‘grads’ to a right-angle, but this hasn’t caught

on. What about using a unit that is about 57.29577951 degrees? For practical purposes it would be a nightmare! And yet that is the natural unit of measure for reasons that we'll now see. In fact, unless you're a surveyor or a navigator you'll be leaving degrees behind and adopting **radians** as the unit of choice.

One way to define angles is in terms of arc lengths. If we have a sector of a circle with unit radius we define the angle as the length of the arc.



Since the circumference of a circle with radius 1 is  $2\pi$ ,  $360^\circ$  is  $2\pi$  radians and  $90^\circ$  is  $\pi/2$ . The special angles in radians are as follows.

<b>degrees</b>	<b>radians</b>
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$

We choose not to evaluate these as decimals. It's easier, and more exact, to leave them as multiples of  $\pi$ . From now on all angles will be in radians and therefore we write:

$$\begin{aligned} \sin(\theta \pm 2\pi) &= \sin \theta, \\ \cos(\theta \pm 2\pi) &= \cos \theta, \\ \tan(\theta \pm 2\pi) &= \tan \theta, \\ \sin(\theta + \pi) &= -\sin \theta, \\ \cos(\theta + \pi) &= -\cos \theta, \\ \tan(\theta + \pi) &= \tan \theta, \\ \sin(\pi/2 - \theta) &= \cos \theta, \\ \cos(\pi/2 - \theta) &= \sin \theta, \\ \tan(\pi/2 - \theta) &= \cot \theta. \end{aligned}$$

## §7.7. The Functions $\sin x$ , $\cos x$ and $\tan x$

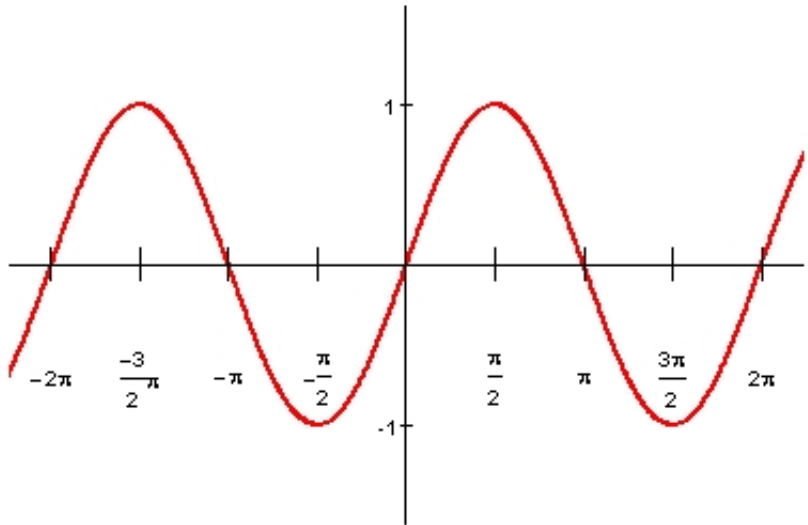
Up till now we've written the sine function as  $\sin \theta$ , or  $\sin A$ . That's because we've thought of  $\theta$  or  $A$  as an angle. Many important applications of the trigonometric functions have nothing to do with angles. So we're now going to think of sine and cosine as applying to real numbers, rather than angles. For this reason we'll usually write  $\sin x$  and  $\cos x$ .

The values of these functions are the values that would apply if  $x$  was an angle measured in radians, so we're not moving too far from what we know already. It's just a different point of view.

If we put our calculator into radian mode (you'll have to read the instruction book to see how this is done for your particular calculator), then enter a number and then press the SIN key you'll get the value of  $\sin x$ .

To check that you can operate your calculator correctly in this way insert 2 and press SIN. You should get approximately 0.9093. (If you get 0.0349 your calculator is in degree mode.) You will have shown that  $\sin 2 = 0.9093$ . One interpretation of this is that the sine of an angle of 2 radians (about  $114^\circ$ ) is 0.9083. But there many other interpretations as there are many uses for the sine function other than something to do with angles.

So try to think of  $\sin x$  in the same way as you think of  $\sqrt{x}$ . What might help is to see the graph of  $y = \sin x$ .



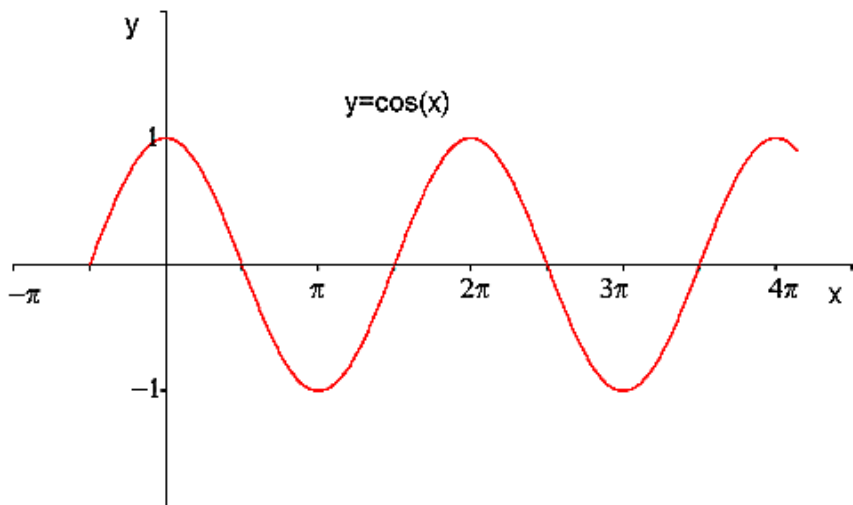
$$y = \sin x$$

Here  $x$  is measured along the  $x$ -axis and so is represented by a length, not an angle.

Notice all the key aspects of  $\sin x$ . When  $x = 0$ ,  $\sin x = 0$ . Then, as  $x$  increases, the value of  $\sin x$  increases until it reaches its maximum at  $x = \pi/2$  (about 1.57). It then starts decreasing and is again 0 when  $x = \pi$ . From  $x = \pi$  to  $x = 2\pi$ , we find that  $\sin x$  is negative. It decreases to a minimum at  $x = 3\pi/2$  and then works its way back to zero. From here it starts its fluctuation all over again.

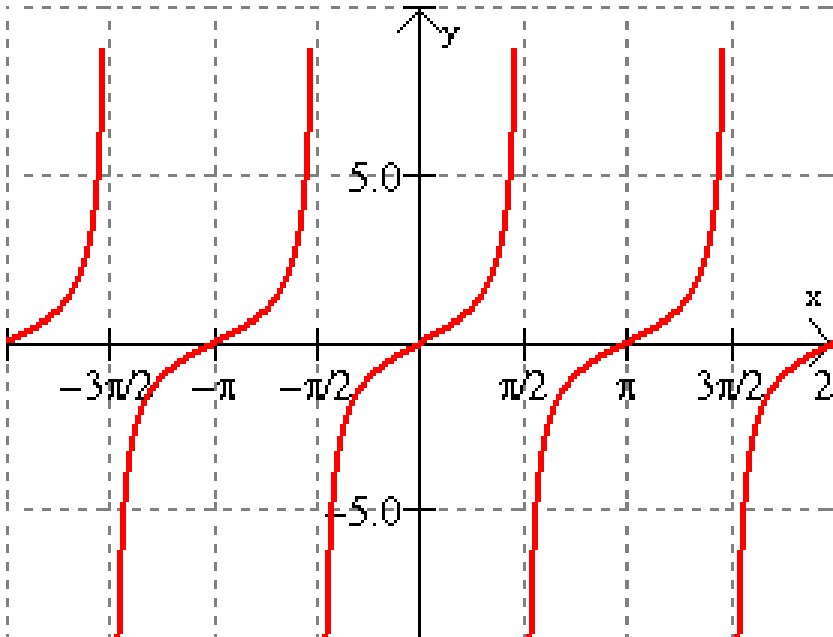
You can clearly see the periodic nature of the sine function. Also there's the fact that the graph looks the same if you turn it upside down (rotating it through 180 degrees about the vertical axis – or should I say  $\pi$  radians!). This is due to the fact that  $\sin(-x) = -\sin x$ .

The graph of  $y = \cos x$  has exactly the same shape. It is just translated along the  $x$ -axis through a distance of  $\pi/2$ . This reflects the fact that  $\cos x = \sin(x + \pi/2)$ .



The graph of  $y = \tan x$  looks quite different. Notice how there are discontinuities at  $x = \pi/2, 3\pi/2$  etc. When I said  $\tan \pi/2 = \infty$  I was only telling half the story. Because, while it's true that  $\tan x$  approaches  $\infty$  as  $x$  approaches  $\pi/2$  from *below*, when  $x$  approaches  $\pi/2$  from above,  $\tan x$  approaches  $-\infty$ .

$$y = \tan x$$



Whenever there's a repetitive action the sine function is likely to be close by. A pendulum on a clock, a piston moving backwards and forwards, these are examples of **simple harmonic motion**. If you've ever seen an oscilloscope (an electronic display that shows sounds as pictures) showing a pure musical tone, you'll see a sine curve.



You may wonder how a calculator can work out sines and cosines, when it doesn't have built-in angles. It does it using just the four basic arithmetic operations. It could use the power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{ and}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

These are infinite series but they converge quite rapidly. This means that only a fairly small number of terms need to be used to get quite good approximations (to eight or ten decimal places). Of course the  $x$  represents angles measured in radians. If it's in degree mode it has to convert to radians first.

In actual practice certain other series are used, ones that are more complicated to describe but which converge even more rapidly. The derivation of these series depends on calculus.